Temporal-Logic Combinators

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What is Temporal Logic?

- A modal logic where propositions are quantified over time.
- Also known as tense logic.
- Useful for reasoning about time-varying values.

What is Time?

- A temporal logic has an underlying time domain.
- The nature of the time domain can vary. E.g.
  - Continuous or Discrete
  - Linear, Cyclic or Branching
  - Finite or Infinite
- But we assume it is a pre-order.

Agda Encoding

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Temporal Predicates

$$TPred \; = \; Time \rightarrow Set$$

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The standard temporal-logic combinators can then be encoded directly in Agda.

Logical Operators

Lifted Operators

$$\_\wedge\_ \;:\; \mathit{TPred} \to \mathit{TPred} \to \mathit{TPred}$$
$$(\varphi \wedge \psi)\; t \;=\; \varphi\; t \times \psi\; t$$

$$\_\vee\_ \;:\; \mathit{TPred} \to \mathit{TPred} \to \mathit{TPred}$$
$$(\varphi \vee \psi)\; t \;=\; \varphi\; t \uplus \psi\; t$$

$$\_\Rightarrow\_ \;:\; \mathit{TPred} \to \mathit{TPred} \to \mathit{TPred}$$
$$(\varphi \Rightarrow \psi)\; t \;=\; \varphi\; t \to \psi\; t$$

$$\bot \;:\; \mathit{TPred}$$
$$\bot\; t \;=\; \mathit{False}$$

$$\top \;:\; \mathit{TPred}$$
$$\top\; t \;=\; \mathit{True}$$

Logical Operators

Defining Negation

$$\neg \; : \; \mathit{TPred} \rightarrow \mathit{TPred}$$
$$\neg\, \varphi \; = \; \varphi \Rightarrow \bot$$

Priorean Operators

First devised by Arthur Prior [Pri67].

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Global

`‘‘At all points in the future’’`

$$\mathbf{G} \; : \; \mathit{TPred} \rightarrow \mathit{TPred}$$
$$\mathbf{G}\; \varphi\; t \;=\; (t' \;:\; \mathit{Time}) \rightarrow (t' > t) \rightarrow \varphi\; t'$$

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History

``At all points in the past''

$$\mathbf{H} \;:\; \mathit{TPred} \to \mathit{TPred}$$
$$\mathbf{H}\, \varphi\, t \;=\; (t' \;:\; \mathit{Time}) \to (t' < t) \to \varphi\, t'$$

Priorean Operators

Future

``At some point in the future''

$$\mathbf{F} \; : \; \mathit{TPred} \rightarrow \mathit{TPred}$$
$$\mathbf{F}\ \varphi\ t \;=\; \Sigma\,[t' : \mathit{Time}]\,((t' > t) \times \varphi\ t')$$

Past

``At some point in the past''

$$\mathbf{P} \; : \; \mathit{TPred} \rightarrow \mathit{TPred}$$
$$\mathbf{P}\ \varphi\ t \;=\; \Sigma\,[t' : \mathit{Time}]\,((t' < t) \times \varphi\ t')$$

$$\textbf{syntax}\ \Sigma\ A\ (\lambda\ a \rightarrow B) \;=\; \Sigma\,[a : A]\ B$$

Reflexive Priorean Operators

$$\mathbf{G^r} \; : \; \mathit{TPred} \rightarrow \mathit{TPred}$$
$$\mathbf{G^r}\ \varphi \; = \; \varphi \wedge \mathbf{G}\ \varphi$$

$$\mathbf{H^r} \; : \; \mathit{TPred} \rightarrow \mathit{TPred}$$
$$\mathbf{H^r}\ \varphi \; = \; \varphi \wedge \mathbf{H}\ \varphi$$

$$\mathbf{F^r} \; : \; \mathit{TPred} \rightarrow \mathit{TPred}$$
$$\mathbf{F^r}\ \varphi \; = \; \varphi \vee \mathbf{F}\ \varphi$$

$$\mathbf{P^r} \; : \; \mathit{TPred} \rightarrow \mathit{TPred}$$
$$\mathbf{P^r}\ \varphi \; = \; \varphi \vee \mathbf{P}\ \varphi$$

Since/Until Operators

There are also more expressive binary operators, such as Since and Until [Bur82].

Since

$\_\mathbf{S}\_ \; : \; \mathit{TPred} \rightarrow \mathit{TPred} \rightarrow \mathit{TPred}$

$\varphi \; \mathbf{S} \; \psi \approx \varphi \; \mathtt{has\ held\ since} \; \psi \; \mathtt{held}$

Until

$\_\mathbf{U}\_ \; : \; \mathit{TPred} \rightarrow \mathit{TPred} \rightarrow \mathit{TPred}$

$\varphi \; \mathbf{U} \; \psi \approx \varphi \; \mathtt{will\ hold\ until} \; \psi \; \mathtt{holds}$

Properties of Time

It is possible to state temporal formulae that hold if and only if the underlying time domain has a specific property [Ven01].

Time-Domain Properties

$$\begin{aligned} \mathit{FirstPoint} &: \mathit{TPred} \\ \mathit{FirstPoint} &= \mathbf{P}^{\mathbf{r}}\ (\mathbf{H}\ \bot) \end{aligned}$$

$$\begin{aligned} \mathit{EndPoint} &: \mathit{TPred} \\ \mathit{EndPoint} &= \mathbf{F}^{\mathbf{r}}\ (\mathbf{G}\ \bot) \end{aligned}$$

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$$\begin{aligned} &\mathit{Density} : \mathit{Set} \\ &\mathit{Density} = (\varphi : \mathit{TPred}) \rightarrow \mathit{Always}\ (\mathbf{F}\ \varphi \Rightarrow \mathbf{F}\ (\mathbf{F}\ \varphi)) \\ &\quad \textbf{where} \\ &\qquad \mathit{Always} : \mathit{TPred} \rightarrow \mathit{Set} \\ &\qquad \mathit{Always}\ \varphi = (t : \mathit{Time}) \rightarrow \varphi\ t \end{aligned}$$

Functional Reactive Programming (FRP)

- FRP is based around time-varying values called signals:

$$Signal\ A \ = \ Time \rightarrow A$$

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- FRP is based around time-varying values called signals:

$$Signal\ A \;=\; Time \rightarrow A$$

- Typically, an FRP time domain:
  - is linear;
  - has a start point;
  - does not have an end point;
  - may be either continuous or discrete.

Functional Reactive Programming (FRP)

Time-Varying Equality

$$\_\doteq\_ \ : \ \mathit{Signal}\ A \rightarrow \mathit{Signal}\ A \rightarrow \mathit{TPred}$$
$$(s_1 \ \doteq \ s_2)\ t \ = \ s_1\ t \equiv s_2\ t$$

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Causality

$$\mathit{Causal} \;:\; (\mathit{Signal}\ A \rightarrow \mathit{Signal}\ B) \rightarrow \mathit{Set}$$
$$\mathit{Causal}\ f \;=\; \forall\,(s_1\ s_2) \rightarrow \mathit{Always}\ (\mathbf{H}^{\mathsf{r}}\ (s_1 \doteq s_2) \Rightarrow f\ s_1 \doteq f\ s_2)$$

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Decoupledness

$$\mathit{Decoupled} \;:\; (\mathit{Signal}\ A \rightarrow \mathit{Signal}\ B) \rightarrow \mathit{Set}$$
$$\mathit{Decoupled}\ f \;=\; \forall\ (s_1\ s_2) \rightarrow \mathit{Always}\ (\mathbf{H}\ (s_1 \doteq s_2) \Rightarrow f\ s_1 \doteq f\ s_2)$$

Summary

- Temporal-logic combinators are a useful notation for expressing and reasoning about time-varying properties.
- They (often) allow definitions to be stated intuitively and concisely.
- See my thesis for more FRP-based examples [Scu11].

References

John P. Burgess.
Axioms for tense logic. I. "Since" and "Until".
Notre Dame Journal of Formal Logic, 23(4):367–374, 1982.

Arthur N. Prior.
Past, Present and Future.
Oxford University Press, 1967.

Neil Sculthorpe.
Towards Safe and Efficient Functional Reactive Programming.
PhD thesis, School of Computer Science, University of Nottingham, 2011.

Yde Venema.
Temporal logic.
In *The Blackwell Guide to Philosophical Logic*, chapter 10, pages 203–223. Blackwell, 2001.